

Propagators in the continuum limit: from molecules to scalar fields

E. Sadurní

Instituto de Física
Departamento de Física Teórica
Universidad Nacional Autónoma de México
Apartado Postal 20-364, 01000 México D.F., México
sadurni at fisica.unam.mx

Abstract

The propagator of linear molecules whose constituents interact through oscillator potentials can be obtained in a closed form for N atoms as long as $N \leq 4$. We compute the propagator for arbitrary N in the approximation $N \gg 1$. Taking advantage of this result it is possible to analyze the limit in which the molecule has an infinite number of constituents with infinitesimal length of separation, corresponding to the quantization of a string, elastic rod or the second quantization of a Klein Gordon particle. The evolution of some specific initial conditions is also studied, namely the time development of states with minimal dispersion and the effect of sudden perturbations on the vacuum of the scalar field theory.

Keywords: Propagators, harmonic oscillator, continuous media, scalar field.

1 Introduction

Transient phenomena in quantum mechanics have been of interest since the developments in [1] a long time ago and others which are more recent [2],

[3]. The treatment of this subject demands a dynamical study of different quantum systems corresponding to specific problems. In view of the simple theoretical results obtained for models involving oscillators in [4], there arises naturally the question of their generalization to systems with infinite degrees of freedom.

In many textbooks [5] we find treatments of continuous media as the limit of interacting discrete systems in the context of classical mechanics. However, it is possible to exploit this analogy in the quantum picture through the Feynman formulation of the propagator [6], connecting the latter with classical lagrangians. In this fashion one may study discretized versions of second order field equations and quantize them before taking the continuum limit. The inverse order of this limits makes it possible to perform certain calculations, while when taken in the usual order gives rise to path integrals of field configurations. The latter may be difficult to handle or may even lack of an appropriate mathematical definition.

Our prototype discrete system will be the linear molecule and we proceed to compute its propagator in section 2. In section 3 the continuum limit of our system is taken calculating thereby the propagator for strings and second quantized scalar fields. In sections 4 and 5 we apply our results to certain initial conditions, including sudden external interactions.

2 Propagator of the linear molecule

We start with a lagrangian in $1 + 1$ dimensions of a chain of N particles of equal masses m connected by equal springs of constant $k = m\omega^2$

$$\mathcal{L} = \frac{m}{2} \sum_{i=1}^N \left(\dot{x}_i^2 - \omega^2 (x_i - x_{i+1})^2 \right) \equiv \sum_{i=1}^N \mathcal{L}_i \quad (1)$$

It is costumary to find the orthogonal transformation which separates the problem in N independent lagrangians (normal modes) and we will do so in the following. Letting $(\mathbf{x})_i = x_i$ and

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & \dots & & -1 & 2 & -1 \\ 0 & \dots & & 0 & -1 & 1 \end{pmatrix}_{N \times N} \quad (2)$$

our lagrangian becomes

$$\mathcal{L} = \frac{m}{2} (\dot{\mathbf{x}}^2 - \omega^2 \mathbf{x}^T \mathbf{V} \mathbf{x}) \quad (3)$$

and it is left to diagonalize \mathbf{V} . It can be shown (see Appendix) that the eigenvalues of \mathbf{V} are given by

$$\lambda_n = 4 \sin^2 \left(\frac{n\pi}{2(N-1)} \right) + O(1/N), \quad 0 \leq n \leq N-1 \quad (4)$$

which is particularly useful for large N . We can see that $N \sim 10$ is already a good approximation as can be checked numerically. We resort to this type of formula since it allows to manipulate our expressions simbolicly though numerical methods to compute λ_n exist. Notice that in the limit $N \rightarrow \infty$ the λ 's (frequencies of the normal modes) populate densely the interval $[0, 4]$ distributed by a trigonometric law. It is straightforward then to write

$$\mathcal{L} = \frac{m}{2} (\dot{\mathbf{y}}^2 - \omega^2 \mathbf{y}^T \mathbf{D} \mathbf{y}) \quad (5)$$

where $\mathbf{y} = \mathbf{O} \mathbf{x}$, \mathbf{O} being the orthogonal transformation corresponding to the diagonalization of \mathbf{V} into $\mathbf{D} = \text{diag} (\lambda_0, \dots, \lambda_{N-1})$. The propagator is written as [7]

$$K(\mathbf{y}, \mathbf{y}'; t) = \prod_{n=0}^{N-1} \sqrt{\frac{m\omega\lambda_n^{1/2}}{2\pi i\hbar \sin(\omega\lambda_n^{1/2}t)}} \times \\ \times \exp \left[\frac{im\omega\lambda_n^{1/2}}{2\hbar \sin(\omega\lambda_n^{1/2}t)} (\cos(\omega\lambda_n^{1/2}t)(y_n^2 + y_n'^2) - 2y_n y_n') \right] \quad (6)$$

and this expression is exact as long as the λ 's contain corrections $O(1/N)$, but these do not alter the functional form of (6). Notice that the factor associated to $n = 0$ corresponds to $\lambda_0 = 0$ and can be written as

$$K_0(\mathbf{y}_0, \mathbf{y}'_0; t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp \left[\frac{im}{2\hbar t} (y_0 - y'_0)^2 \right] \quad (7)$$

which is the propagator of the free particle, finding thereby that y_0 is the coordinate of the center of mass. The meaningful degrees of freedom are therefore indicated by \mathbf{y} , but we can at any moment employ the inverse transformation $\mathbf{x} = \mathbf{O}^T \mathbf{y}$ since \mathbf{O} can be written explicitly (see Appendix).

3 The continuum limit of the linear molecule propagator

Here we will consider the continuum limit as that indicated in [5] or [8]. First of all let us recall that the lagrangian (1) can be multiplied by a parameter a with the dimensions of length so that

$$\mathcal{L} = \sum_{i=1}^N a \left(\frac{m}{2a} \dot{x}_i^2 - \frac{m\omega^2 a}{2} \left(\frac{x_i - x_{i+1}}{a} \right)^2 \right) \quad (8)$$

$$\xrightarrow[N \rightarrow \infty, \quad a \rightarrow 0]{} \int_0^L d\xi \frac{\mu}{2} (\partial_t \phi)^2 - \frac{\mu \Omega^2}{2} (\partial_\xi \phi)^2 \quad (9)$$

as long as $x_i(t) \rightarrow \phi(\xi, t)$, $a = \frac{L}{N} \rightarrow d\xi$, $\omega a \rightarrow \Omega$. The constant L is interpreted as the length of the resulting string and Ω is such that $\mu \Omega^2 = Y$, *i.e.* Young's module.

For the propagator limit we use Feynman's prescription

$$\begin{aligned} K(\mathbf{x}, \mathbf{x}'; t) &= \left(\prod_{i=1}^N \int_{x_i(0)=x_0}^{x_i(t)=x_f} Dx_i(\cdot) \right) \exp \left(-\frac{i}{\hbar} \int_0^t d\tau \sum_{i=1}^N \mathcal{L}_i \right) \\ &\xrightarrow[N \rightarrow \infty]{} \int_{\phi(x_0, 0)}^{\phi(x_f, t)} D\phi(\cdot) \exp \left(\frac{-i}{\hbar} \int_0^t \int_0^L d\tau d\xi \frac{\mu}{2} (\partial_t \phi)^2 - \frac{\mu \omega^2}{2} (\partial_\xi \phi)^2 \right) \end{aligned} \quad (10)$$

which is the propagator of a string, elastic rod or a massless scalar relativistic particle when the classical length $L \rightarrow \infty$ and the identification $\Omega \leftrightarrow c$ takes place.

Though we are quite familiar with this form of the limit, in normal coordinates \mathbf{y} some subtleties arise. First of all, we notice that the $N \rightarrow \infty$ limit of lagrangian (5) is equal to the limit of the following lagrangian

$$\sum_{j=1}^N a \left(\frac{\mu}{2} \dot{y}_j^2 - \frac{\mu \Omega^2}{2} (\lambda_j/a^2) y_j^2 \right) \quad (11)$$

where we can observe that the eigenvalues are now λ_j/a^2 . Since

$$\frac{\lambda_j}{a^2} = \frac{4N^2}{L^2} \sin^2 \left(\frac{j\pi}{2(N-1)} \right) + O(1/N) \xrightarrow{N \rightarrow \infty} \left(\frac{j\pi}{L} \right)^2 \equiv \Lambda_j \quad (12)$$

we see that these eigenvalues are distributed discretely for finite j by a square law, just as the energy spectrum of a string with given initial and final conditions at a fixed time. Therefore, the limit of (11) is not an integral, but a series and it will make sense if we set $\sqrt{a}y_j \rightarrow \eta_j$, *i.e.* if fields are rescaled. To this effect we should also rescale the orthogonal operator \mathbf{O} in the following way

$$\frac{1}{\sqrt{a}} O_{ij} \xrightarrow{N \rightarrow \infty} O_j(\xi) \quad (13)$$

so that the transformations between fields become

$$\begin{aligned} \eta_j &= \int_0^L d\xi O_j(\xi) \phi(\xi), & \phi(\xi) &= \sum_j O_j(\xi) \eta_j \\ \int_0^L O_j(\xi) O_l(\xi) d\xi &= \delta_{jl}, & \sum_j O_j(\xi) O_j(\xi') &= \delta(\xi - \xi') \end{aligned} \quad (14)$$

where $a^{-1} \delta_{ii'} \rightarrow \delta(\xi - \xi')$. With all these considerations, lagrangian (5) possesses the limit

$$\mathcal{L} \rightarrow \sum_{j=0}^{\infty} \frac{\mu}{2} \dot{\eta}_j^2 - \frac{\mu \Omega^2}{2} (j\pi/L)^2 \eta_j^2 \quad (15)$$

whose discrete form stems from the discreteness of the eigenvalues Λ_j as we have proven.

The propagator related to lagrangian (15) can be written directly or it can be computed as the limit $N \rightarrow \infty$ of propagator (6) but taking into account that the prefactor is proportional to $m^{N/2} \sim a^{N/2} \mu^{N/2}$. Thus, for the

fields η_j the propagator is multiplied by the jacobian $a^{-N/2}$, cancelling the scale factor emerging from the mass. The resulting expression is

$$K_{\text{string}}(\eta, \eta'; t) = f(t) \exp \left(\frac{i\mu\Omega}{2\hbar} \sum_{j=0}^{\infty} \frac{j\pi}{L \sin(\Omega j\pi t/L)} \left[(\eta_j^2 + \eta_j'^2) \cos(\Omega j\pi t/L) - 2\eta_j \eta_j' \right] \right) \quad (16)$$

with

$$f(t) \equiv \prod_{j=0}^{\infty} \frac{\mu\Omega j\pi}{2\pi i\hbar L \sin(\omega j\pi t/L)} \quad (17)$$

The definition for f deserves some comments. The convergence of this infinite product can be spoiled by the presence of μ and Ω , leading to $f = \infty, 1$ or 0 for different values of these parameters. Nevertheless the propagator itself obeys a normalization condition even when the prefactor is a divergent or a vanishing quantity. This stems from the fact that K is the representation of a unitary operator and preserves norm. The limit of propagators should be taken under the integral $\int D\phi[\cdot]$ or its discrete version from which it is easy to see that a rescaling of fields $\bar{\phi} = \sqrt{\frac{\mu\Omega}{2\pi\hbar}}\phi$, $\bar{\eta} = \sqrt{\frac{\mu\Omega}{2\pi\hbar}}\eta$ gets rid of the problem. Using dimensionless fields $\bar{\eta}$ is the price to be paid. With all this the propagator becomes

$$K_{\text{string}}(\bar{\eta}, \bar{\eta}'; t) = \bar{f}(t) \exp \left(i \sum_{j=0}^{\infty} \frac{j\pi^2}{L \sin(\Omega j\pi t/L)} \left[(\bar{\eta}_j^2 + \bar{\eta}_j'^2) \cos(\Omega j\pi t/L) - 2\bar{\eta}_j \bar{\eta}_j' \right] \right) \quad (18)$$

with

$$\bar{f}(t) \equiv \prod_{j=0}^{\infty} \frac{j\pi/L}{i \sin(\omega j\pi t/L)} = \sqrt{\det \left[i(V + i\epsilon) (\sin(V + i\epsilon))^{-1} \right]} \quad (19)$$

$V \equiv V_{\infty}$ and ϵ arbitrarily small. Let us now write the propagator of a second-quantised Klein Gordon particle of mass M . First of all we need to add a mass term $\frac{mM^2c^2}{2\hbar} \mathbf{x} \cdot \mathbf{x}$ to lagrangian (1). Since this term is invariant under the orthogonal transformation with associated matrix \mathbf{O} , it is sufficient to replace ω in (6) by $\omega_j = \sqrt{\omega^2 + \frac{M^2c^4}{\hbar^2\lambda_j}}$. As we have indicated, we must set

$\Omega = c$. Once this is done we need to extend the molecule to the whole real line by taking $L \rightarrow \infty$. We identify the continuous wave number as $j\pi/L \rightarrow k$, giving a differential $\pi/L \rightarrow dk$. The fields $\bar{\eta}$ must be rescaled again in order to have the limit $\sqrt{L}\bar{\eta}_j \rightarrow \eta(k)$. The orthogonal transformation obeys $\sqrt{L}O_j(\xi) \rightarrow O(\xi, k)$ in order to have field transformations in full integral form, *i.e.*

$$\begin{aligned} \eta(k) &= \int_{-\infty}^{\infty} d\xi O(k, \xi) \phi(\xi), & \phi(\xi) &= \int_{-\infty}^{\infty} dk O(\xi, k) \eta(k) \\ \int_{-\infty}^{\infty} d\xi O(k, \xi) O(k', \xi) &= \delta(k - k'), & \int_{-\infty}^{\infty} dk O(\xi, k) O(\xi', k) &= \delta(\xi - \xi') \end{aligned} \quad (20)$$

where $L\delta_{jj'} \rightarrow \delta(k - k')$. Meanwhile, the lagrangian (15) with a mass term has the following limit when $L \rightarrow \infty$

$$\int_{-\infty}^{\infty} dk |\dot{\eta}(k)|^2 - (E_k/\hbar)^2 |\eta(k)|^2 \quad (21)$$

where we have computed

$$\hbar \lambda_j^{1/2} \omega_j \xrightarrow{L \rightarrow \infty} \sqrt{\hbar^2 c^2 k^2 + M^2 c^4} \equiv E_k \quad (22)$$

From the equations of motion induced by this lagrangian we see that spatial derivatives become factors of k , inferring thus that $O(k, \xi)$ is the Fourier kernel. Finally the sought propagator is given by

$$K_{KG}[\eta, \eta', t] = g(t) \exp \left[\int_{-\infty}^{\infty} dk \frac{iE_k}{\hbar^2 k \sin\left(\frac{E_k t}{\hbar}\right)} \left[\cos\left(\frac{E_k t}{\hbar}\right) (|\eta(k)|^2 + |\eta'(k)|^2) - 2\eta(k)\eta'(k) \right] \right] \quad (23)$$

with

$$g(t) \equiv \prod_{k \in \mathbf{R}} \sqrt{\frac{E_k}{i\hbar \sin\left(\frac{E_k t}{\hbar}\right)}} \quad (24)$$

The prefactor is given again in terms of an infinite product and it is left left indicated without discussing its convergence. At any rate, it seems more suitable to work with the molecule model and its propagator (6) or its *massive* version in order to study dynamical effects on strings or particle fields.

Any meaningful time dependent quantity of these systems will be treated discretely and limits will be computed at the end of calculations.

4 Minimally localized molecule and its continuum limits

At this point we are ready to apply propagator (6) to some initial conditions. Consider the linear molecule of N atoms to be in a state which is minimally dispersed, *i.e.* a gaussian distribution in space for each atom:

$$\psi(\mathbf{x}', 0) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/4} \exp\left[-\frac{1}{4\sigma^2} \sum_{i=0}^{N-1} x_i'^2\right] \quad (25)$$

where the width σ is the same for all components. This wave function has the property

$$\psi(\mathbf{x}', 0) = \psi(\mathbf{Ox}', 0) = \psi(\mathbf{y}', 0). \quad (26)$$

and it describes a localization in phase space with minimal uncertainty. In the continuum limit $N \rightarrow \infty$, initial condition (25) becomes

$$\begin{aligned} \psi[\phi, 0] &= \lim_{N \rightarrow \infty} \left(\frac{1}{2\pi\sigma^2}\right)^{N/4} \exp\left(-\frac{1}{4s^2} \int_0^L d\xi \phi^2(\xi)\right) \\ &= \lim_{N \rightarrow \infty} \left(\frac{1}{2\pi\sigma^2}\right)^{N/4} \exp\left(-\frac{1}{4s^2} \sum_{j=0}^{\infty} \eta_j^2\right) \end{aligned} \quad (27)$$

where $\sigma^2 a \rightarrow s^2$. Similar remarks to those following (17) hold here. Despite the limit of the factor preceding the exponential may be vanishing or divergent, functional (27) is normalized and for all t we have $\int D\phi |\psi[\phi, t]|^2 = 1$. Thus, functional (27) is a probability amplitude depending on field configurations. In order to obtain a normalized functional in normal coordinates we need though to use the jacobian $a^{-N/2}$ to write the initial condition as

$$\tilde{\psi}[\eta, 0] = \left[\lim_{N \rightarrow \infty} \left(\frac{1}{2\pi s^2}\right)^{N/4}\right] \exp\left(-\frac{1}{4s^2} \sum_{j=0}^{\infty} \eta_j^2\right) \quad (28)$$

so that

$$\int \prod_j d\eta_j |\tilde{\psi}[\eta, 0]|^2 = 1 \quad (29)$$

Now we apply (6) to (26) by integrating over \mathbf{y}' to obtain

$$\psi(\mathbf{y}, t) = \left(\prod_{j=0}^{N-1} \left(\frac{1}{2\pi\sigma_j^2(t)} \right)^{1/4} \right) \exp \left[- \sum_{j=0}^{N-1} \frac{y_j^2}{4\sigma_j^2(t)} \right] e^{i\Delta(t)} \quad (30)$$

with

$$\sigma_j(t) = \sigma \sqrt{1 + \left(\frac{\hbar^2}{2\omega^2 m^2 \sigma^4 \lambda_j} - 1 \right) \sin^2(\omega \lambda_j^{1/2} t)} \quad (31)$$

and $\Delta(t)$ a phase which is irrelevant to the probability density:

$$|\psi(\mathbf{y}, t)|^2 = \left(\prod_{i=0}^{N-1} \left(\frac{1}{2\pi\sigma_i^2(t)} \right)^{1/2} \right) \exp \left[- \sum_{i=0}^{N-1} \frac{y_i^2}{2\sigma_i^2(t)} \right] \quad (32)$$

We observe therefore an oscillatory behaviour of the density widths and amplitudes depending on the corresponding frequencies $\omega \lambda_j^{1/2}$. The center of mass ($\lambda_0 = 0$) distribution exhibits disipation as expected. In the continuum limit (31) yields

$$\sigma_j(t) \xrightarrow{N \rightarrow \infty} s_j(t) = s \sqrt{1 + \left(\frac{\hbar^2 L^2}{2\Omega^2 \mu^2 s^4 j^2 \pi^2} - 1 \right) \sin^2(\Omega t j \pi / L)} \quad (33)$$

whose maximum value decreases with j . For the massive Klein Gordon particle we find a different limit. The discrete case yields

$$\sigma_j(t) = \sigma \sqrt{1 + \left(\frac{\hbar^2}{2\omega_j^2 m^2 \sigma^4 \lambda_j} - 1 \right) \sin^2(\omega_j \lambda_j^{1/2} t)} \quad (34)$$

Taking $N \rightarrow \infty$, $L \rightarrow \infty$, initial condition (28) becomes

$$\tilde{\psi}[\eta, 0] = \left[\lim_{N \rightarrow \infty} \left(\frac{1}{2\pi \bar{s}^2} \right)^{N/4} \right] \exp \left(-\frac{1}{4\bar{s}^2} \int_{-\infty}^{\infty} dk \eta^2(k) \right) \quad (35)$$

after rescaling widths $\bar{s} = \sqrt{\frac{\mu\Omega}{2\hbar}} s$. The wave functional for fields $\eta(k)$ at time t is now

$$\tilde{\psi}[\eta, t] = \left[\lim_{N \rightarrow \infty} \left(\frac{1}{2\pi s_k^2(t)} \right) \right] \exp \left(-\int_{-\infty}^{\infty} dk \frac{|\eta(k)|^2}{4s_k^2(t)} \right) \quad (36)$$

with

$$s_k^2(t) = \bar{s} \sqrt{1 + \left(\frac{2\hbar^2 c^2}{E_k^2 \bar{s}^4} - 1 \right) \sin \left(\frac{E_k t}{\hbar} \right)} \quad (37)$$

This quantity is again oscillatory but with a frequency given by the relativistic formula of kinetic energy. The most probable field configuration remains $\eta \equiv 0$, but the field localization width can spread to a factor $(\sqrt{2}\hbar c)/(E_k \bar{s}^2)$.

To illustrate the significance of this result let us compute the average number of quanta for each mode using states $\psi(\mathbf{y}, t)$. In the discrete case we have

$$\langle N_j \rangle = \left\langle (\hbar\omega\lambda_j^{1/2})^{-1} H_{\text{oscillator}} \left(-i\frac{\partial}{\partial y_j}, y_j, \omega\lambda_j^{1/2} \right) - 1/2 \right\rangle \quad (38)$$

$$= \frac{m\omega}{2\hbar} \sigma_j^2(t) \lambda_j^{1/2} + \frac{3\hbar}{4m\omega} (\sigma_j^2(t) \lambda_j^{1/2})^{-1} - \frac{1}{2} \quad (39)$$

Taking the continuum limit $N \rightarrow \infty$ the average number results in

$$\langle N_j \rangle = \frac{\mu\Omega}{2\hbar} s_j^2(t) \left(\frac{j\pi}{L} \right) + \frac{3\hbar}{4\mu\Omega} (s_j^2(t) \left(\frac{j\pi}{L} \right))^{-1} - \frac{1}{2} \quad (40)$$

with $s_j(t)$ given by (33). For the massive relativistic particle we have to replace the frequencies by ω_j and take $L \rightarrow \infty$. The expectation values computed by integrating with respect to fields $\eta(k)$ are now multiplied by the squared jacobian $1/L \rightarrow dk/\pi$, thus we get

$$\frac{1}{L} \langle N_j \rangle \xrightarrow{L \rightarrow \infty} \left[\frac{E_k}{\hbar} s_k^2(t) + \frac{3\hbar}{8E_k} (s_k^2(t))^{-1} - \frac{1}{2} \right] \frac{dk}{\pi} \equiv \nu_k(t) dk \quad (41)$$

where we have obtained a number density $\nu_k(t)$ in momentum space as a function of time.

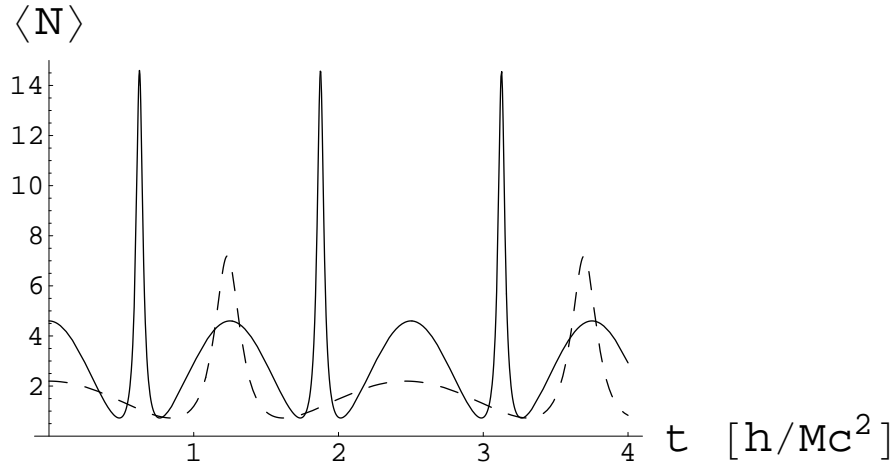


Figure 1: Average number density of particles as a function of t for $k = \frac{10Mc}{\hbar}$ (Solid line) and $k = \frac{5Mc}{\hbar}$ (Dashed line).

An oscillatory density number is not a surprising result when viewed as the evolution of harmonic oscillator number of quanta. However, it is interesting to notice that such density number corresponds to a second-quantised *free* field which has been set initially in the most localized configuration in phase space (gaussian distributions).

5 The string under a sudden electrostatic field and the relativistic particle with a source

Now we may think of another example which involves a scalar field and a sudden interaction with a source which is equivalent to a charged string between the plates of a condensator in the discrete case. Consider a system whose lagrangian is given by

$$\mathcal{L}' = \mathcal{L} + \theta(t)\mathcal{E} \cdot \mathbf{x} \quad (42)$$

where \mathcal{L} is given by (1), $\theta(t)$ is the step function and \mathcal{E} denotes some electrostatic field multiplied by the charge of the i th particle in the molecule, *i.e.* $\mathcal{E}_i = q_i E_i$ (remember that this is a one dimensional example and vectors here stand for the N components of our system). If we set $\frac{q_i}{a} \rightarrow \rho(\xi)$ as the linear charge density and $\frac{q_i E_i}{a} \rightarrow \rho(\xi)E(\xi) \equiv \mathcal{E}(\xi)$, then the limit of (42) is well known

$$\mathcal{L}' \xrightarrow{N \rightarrow \infty} \int_0^L d\xi \left(\frac{\mu}{2} (\partial_t \phi(\xi, t))^2 - \frac{\mu \Omega^2}{2} (\partial_\xi \phi(\xi, t))^2 + \theta(t) \mathcal{E}(\xi) \phi(\xi, t) \right) \quad (43)$$

and leads to equations of motion [8]

$$\partial_t^2 \phi - \Omega^2 \partial_\xi^2 \phi = J(\xi), \quad t > 0 \quad (44)$$

where we have defined a current $J(\xi) = \frac{2}{\mu} \mathcal{E}(\xi)$. But before taking the continuum limit, let us assume that for $t < 0$ the discrete system is in its ground state, *i.e.*

$$\psi(\mathbf{y}, 0) = \prod_{i=0}^{N-1} \left(\left(\frac{\hbar}{\pi m \omega \lambda_i^{1/2}} \right)^{1/4} \exp \left[-\frac{m \omega \lambda_i^{1/2}}{2 \hbar} y_i^2 \right] \right) = |0, \dots, 0\rangle \quad (45)$$

given, of course, in normal coordinates. The source term in (42) can be written as

$$\mathcal{E} \cdot \mathbf{x} = \mathcal{E}' \cdot \mathbf{y}, \quad \mathcal{E}' = \mathbf{O} \mathcal{E} \quad (46)$$

With this orthogonal transformation of the external field and for positive times we can immediately write

$$\mathcal{L}'(\dot{\mathbf{y}}, \mathbf{y}) = \mathcal{L}(\dot{\mathbf{z}}, \mathbf{z}) + \sum_{i=0}^{N-1} \frac{\mathcal{E}_i'^2}{2 m \omega_i^2 \lambda_i} \quad (47)$$

where $z_i \equiv y_i - \frac{\mathcal{E}'_i}{m\omega_i^2\lambda_i}$. In the $N \rightarrow \infty$ limit we must set $\mathcal{E}'_j/\sqrt{a} \rightarrow F_j$ so that

$$F_j = \int_0^L d\xi O_j(\xi) \mathcal{E}(\xi) \quad (48)$$

The propagator related to lagrangian (47) has the form (6) but evaluated in \mathbf{z} and with an additional (but irrelevant) phase factor containing the field energy.

It is quite easy to apply such propagator to the vacuum state (45) and we write the resulting probability density computed in [4] for each mode

$$|\psi_j(\mathbf{y}, t)|^2 = \left(\frac{\pi\hbar}{m\omega\lambda_j^{1/2}} \right)^{1/2} \exp \left[-\frac{m\omega\lambda_j^{1/2}}{\hbar} (y_j + a_j(t))^2 \right] \quad (49)$$

where we can see a time dependent average of the distribution given by

$$a_j(t) = \frac{\mathcal{E}'_j \sin^2(\omega\lambda_j^{1/2}t/2)}{m\omega^2\lambda_j}. \quad (50)$$

The continuum limit yields

$$a_j(t)/\sqrt{a} \longrightarrow \frac{F_j \sin^2(\Omega t j \pi / (2L))}{\mu \Omega^2 (j \pi / L)^2} \equiv \alpha_j(t) \quad (51)$$

For the Klein Gordon Field we again rescale fields and set, therefore, $\sqrt{L}F_j \rightarrow F(k)$ so that $F(k)$ can be the Fourier transform of $\mathcal{E}(\xi)$ when $L \rightarrow \infty$. The average field configuration (51) in the massless case yields

$$\sqrt{\frac{\mu\Omega L}{2\hbar}} \alpha_j(t) \xrightarrow{L \rightarrow \infty} \alpha(k, t) = I(k) \frac{\sin^2(ckt)}{2c^2k} \quad (52)$$

where $I(k)$ is the Fourier transform of $J(\xi)$. As before, the massive case is obtained by replacing the frequencies and the result is

$$\alpha(k, t) = \frac{\hbar^2 I(k) \sin^2\left(\frac{E_k t}{2\hbar}\right)}{2E_k^2} \quad (53)$$

The average number taken between states (49) yields, in the discrete case, a number density in momentum space after taking the limits $N \rightarrow \infty$, $L \rightarrow \infty$. Proceeding analogously to the derivation of (41) we arrive at

$$\frac{1}{L} \langle N_j \rangle \xrightarrow{L \rightarrow \infty} \frac{\hbar^3 |I(k)|^2 \sin^4 \left(\frac{E_k t}{2\hbar} \right)}{4E_k^2} dk \equiv \nu_k(t) dk \quad (54)$$

In this result we observe that the Fourier transform of the current ($I(k)$) determines whether the number density ν increases with energy or not, depending on the explicit form of $J(\xi)$. Another unexpected feature is that even when the source is constant in time (say $t > 0$), the number oscillates with a frequency depending on the relativistic energy of the produced particles.

6 Conclusions

Discrete systems are useful to study quantum dynamical problems whose continuum limits may be difficult to formulate in quantum field theory. We have outlined a method by which we can obtain propagators (6,18,23), functionals (30,32, 49) and physical quantities (40,41,54) by developing some simple examples: molecules, strings (or elastic rods) and Klein Gordon particles in second quantization. The method allows interactions in a non perturbative regime, but their form is kept simple so that lagrangians remain quadratic in their discrete version.

Appendix

Now we proceed to derive expression (4). Consider the matrix \mathbf{V}_N as in (2) and let

$$\mathbf{M}_N = \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ . & & & & & \\ . & & & & & \\ 0 & \dots & & -1 & 2 & -1 \\ 0 & \dots & & 0 & -1 & 2 \end{pmatrix}_{N \times N} \quad (55)$$

Define also the polynomials

$$\phi_N(\lambda) = |\mathbf{V}_N - \lambda \mathbf{I}_N|, \quad \chi_N(\lambda) = |\mathbf{M}_N - \lambda \mathbf{I}_N| \quad (56)$$

It can be easily verified that the following recursion relations hold

$$\begin{pmatrix} \phi_N(\lambda) \\ \chi_N(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -\lambda \\ 1 & 1 - \lambda \end{pmatrix} \begin{pmatrix} \phi_{N-1}(\lambda) \\ \chi_{N-1}(\lambda) \end{pmatrix} \quad (57)$$

The matrix appearing in the RHS of (57) can be diagonalized and then the recursion is solved by computing the powers of that matrix. Let $a = \frac{1}{2}(\lambda + \sqrt{\lambda(\lambda - 4)})$ and choose as initial conditions ϕ_2, χ_2 which are given by

$$\phi_2(\lambda) = \lambda^2 - 2\lambda, \quad \chi_2(\lambda) = \lambda^2 - 3\lambda + 1 \quad (58)$$

Following the procedure indicated in the last paragraph by finding the appropriate similarity transformation we can write (57) as

$$\begin{pmatrix} \phi_N(\lambda) \\ \chi_N(\lambda) \end{pmatrix} = (\lambda - a^2)^{-1} \begin{pmatrix} 1 & 1 \\ \frac{a}{\lambda} & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} (1 - a)^{N-2} & 0 \\ 0 & (1 - \frac{\lambda}{a})^{N-2} \end{pmatrix} \times \\ \times \begin{pmatrix} \lambda & -a\lambda \\ -a^2 & a\lambda \end{pmatrix} \begin{pmatrix} \phi_2(\lambda) \\ \chi_2(\lambda) \end{pmatrix} \quad (59)$$

from which the first component is used to write the secular equation $\phi_N(\lambda) = 0$. We notice that the reality of λ implies $a = e^{i\alpha}$ and therefore $\lambda = 4\cos^2(\alpha/2)$. After some algebraic steps we find that the secular equation is equivalent to

$$(\phi_2(\lambda) - 2\chi_2(\lambda)) \sin((N-2)\alpha) \cos(\alpha/2) - \phi_2(\lambda) \cos((N-2)\alpha) \sin(\alpha/2) = 0 \quad (60)$$

Now we can estimate the solutions of (60) by making the Ansatz

$$\alpha = \frac{N-1-n}{N-1}\pi \quad (61)$$

for finite n . Replacing back in (60) and neglecting terms $O(1/N)$ we see that the secular equation is solved. Therefore

$$\lambda = 4 \sin^2 \left(\frac{n\pi}{2(N-1)} \right) + O(1/N) \quad (62)$$

We can also write the equations for the eigenvectors in recurrence form and these can be solved in terms of the eigenvalues given above.

References

- [1] Moshinsky M *Phys. Rev.* **88** (1952) 625.; Moshinsky M *Am. J. Phys.* **44** (1976) 1037.
- [2] Scheitler G and Kleber M *Z. Phys. D* **9** (1988) 267.
- [3] Kramer T and Moshinsky M *Rev. Mex. Fís.* **51** (4)(2005) 407-414.
- [4] Moshinsky M and Sadurní E 2005 *Proc. Symmetry, Integrability and Geometry: Methods and Applications* SIGMA1(2005)003.
- [5] Goldstein H 1980 *Classical Mechanics* (Addison-Wesley)
- [6] Feynman R and Hibbs A 1965 *Quantum Mechanics and Path Integrals* (McGraw-Hill)
- [7] Grosche C and Steiner F 1998 *Handbook of Feynman path integrals (Springer Tracts in Modern Physics vol 145)* (Springer Berlin).
- [8] Kaku M 1993 *Quantum Field Theory: A Modern Introduction* (Oxford Univesity Press)